Fuzzy Inference Systems

- Fuzzy rules
- Mamdani systems
- Takagi-Sugeno systems

Fuzzy systems

- Fuzzy systems manipulate fuzzy sets to model the world
- Most fuzzy systems are rule based

antecedents linguistic variable fuzzy set

If *water temperature* is **low** and *flow rate* is **low**, then open the warm water tap **slightly**

consequent

Antecedents describe a condition in which a rule is valid locally

Consequent can be a fuzzy set, a number or another model (e.g. a linear model)

Linguistic variable

A numerical variable takes numerical values:

Age = 65

A linguistic variables takes linguistic values:

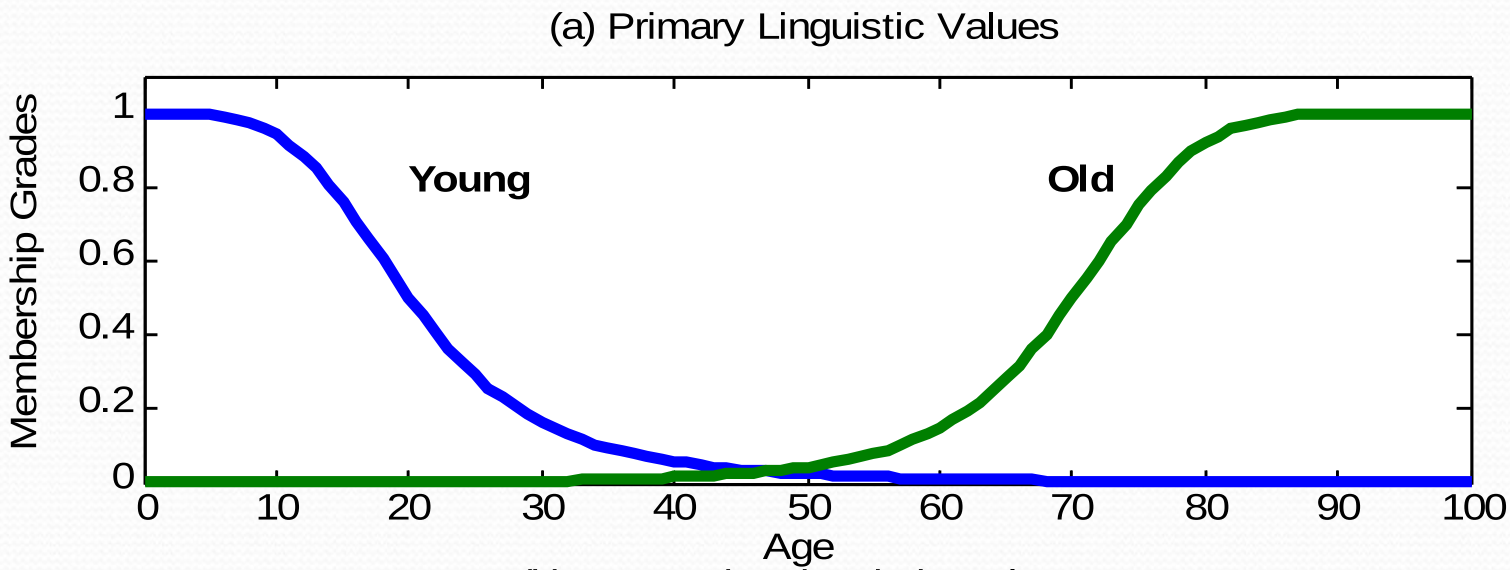
Age is old

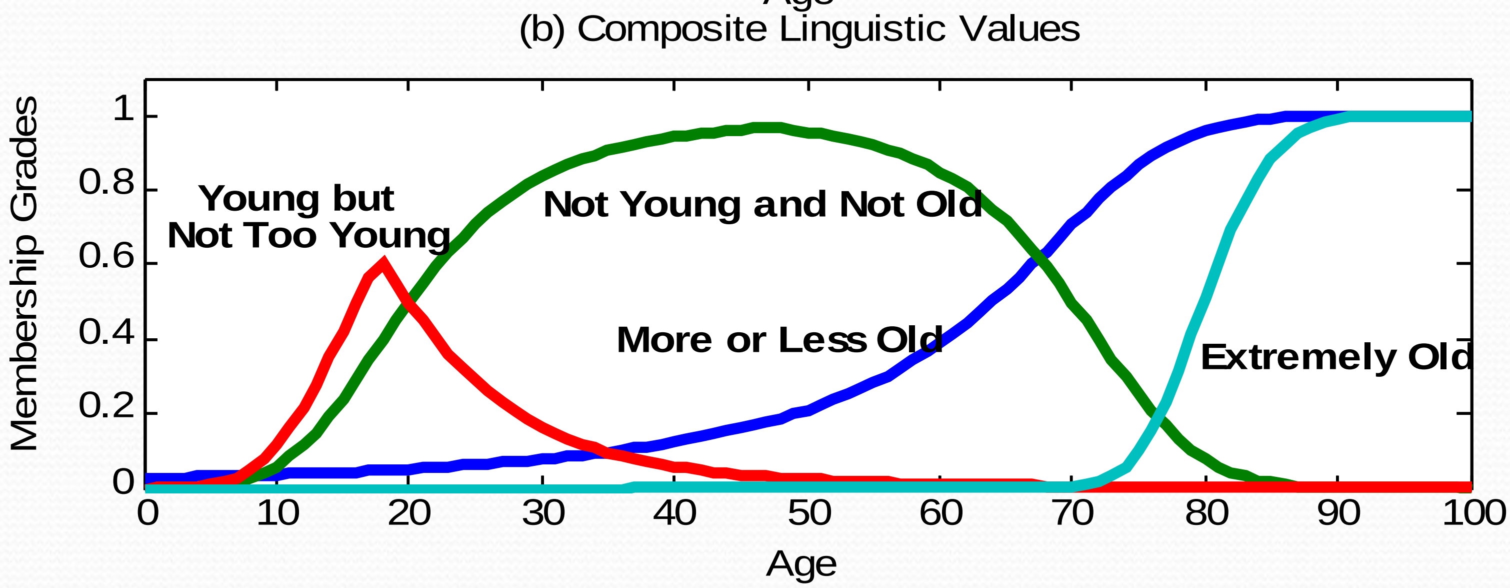
A linguistic value is a fuzzy set.

All linguistic values form a term set:

T(age) = {young, not young, very young,
middle aged, not middle aged,
old, not old, very old, more or less old,
not very young and not very old, ...}

Linguistic values (terms)





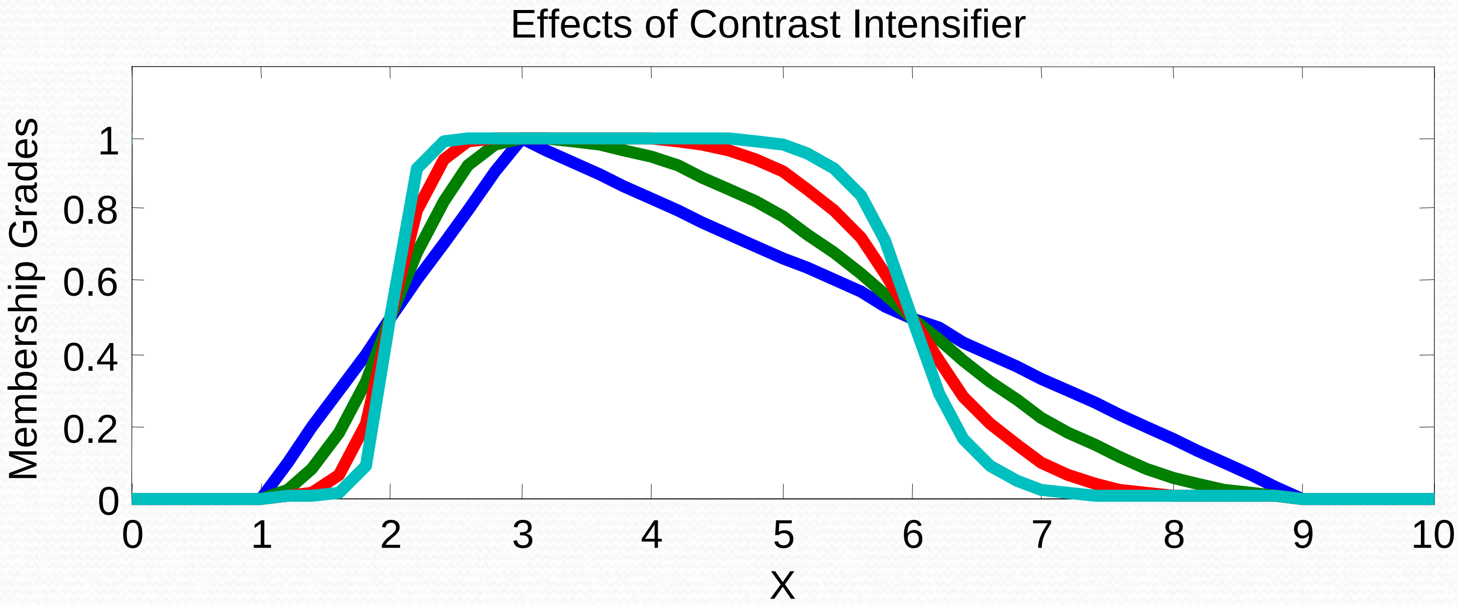
Operations on linguistic values

Concentration: → $CON(A) = A^2$

Dilation: → $DIL(A) = A^{0.5}$

Contrast intensification: →

$$INT(A) = \begin{cases} 2A^2, & 0 \le \mu_A(x) \le 0.5 \\ \neg 2(\neg A)^2, & 0.5 \le \mu_A(x) \le 1 \end{cases}$$



Fuzzy if-then rules

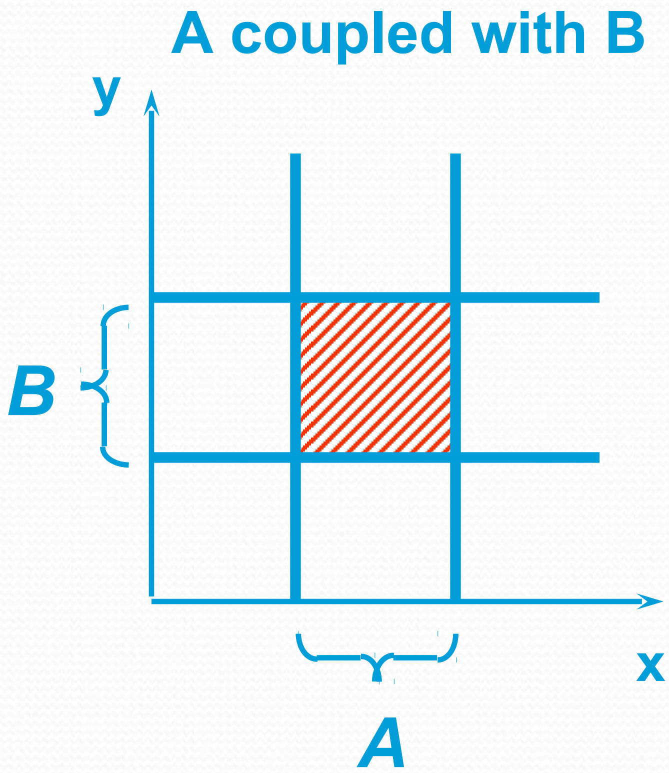
- General format:

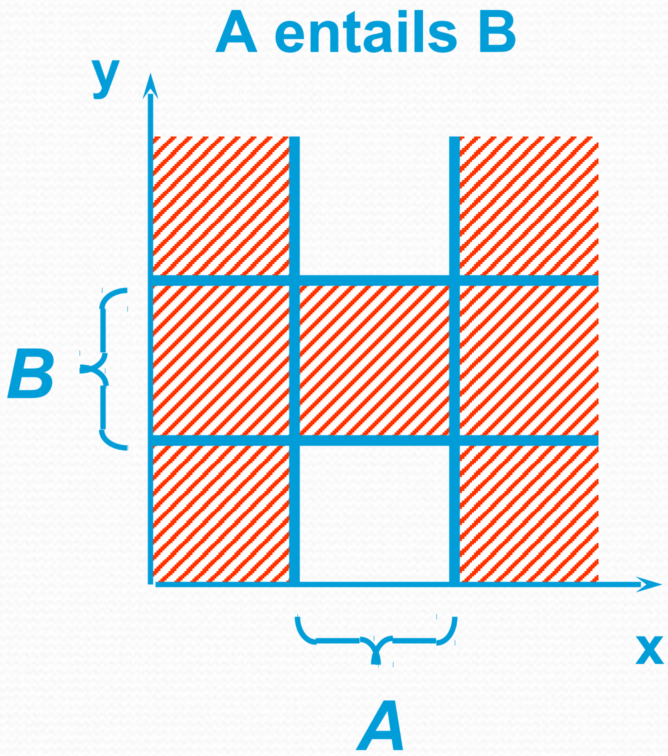
  If x is A then y is B

- Examples:
  - If pressure is high, then volume is small
  - If restaurant is expensive, then order small dishes
  - If a tomato is red, then it is ripe
  - If the speed is high, then apply the brake a little

Interpretation

Two ways to interpret "If x is A then y is B":

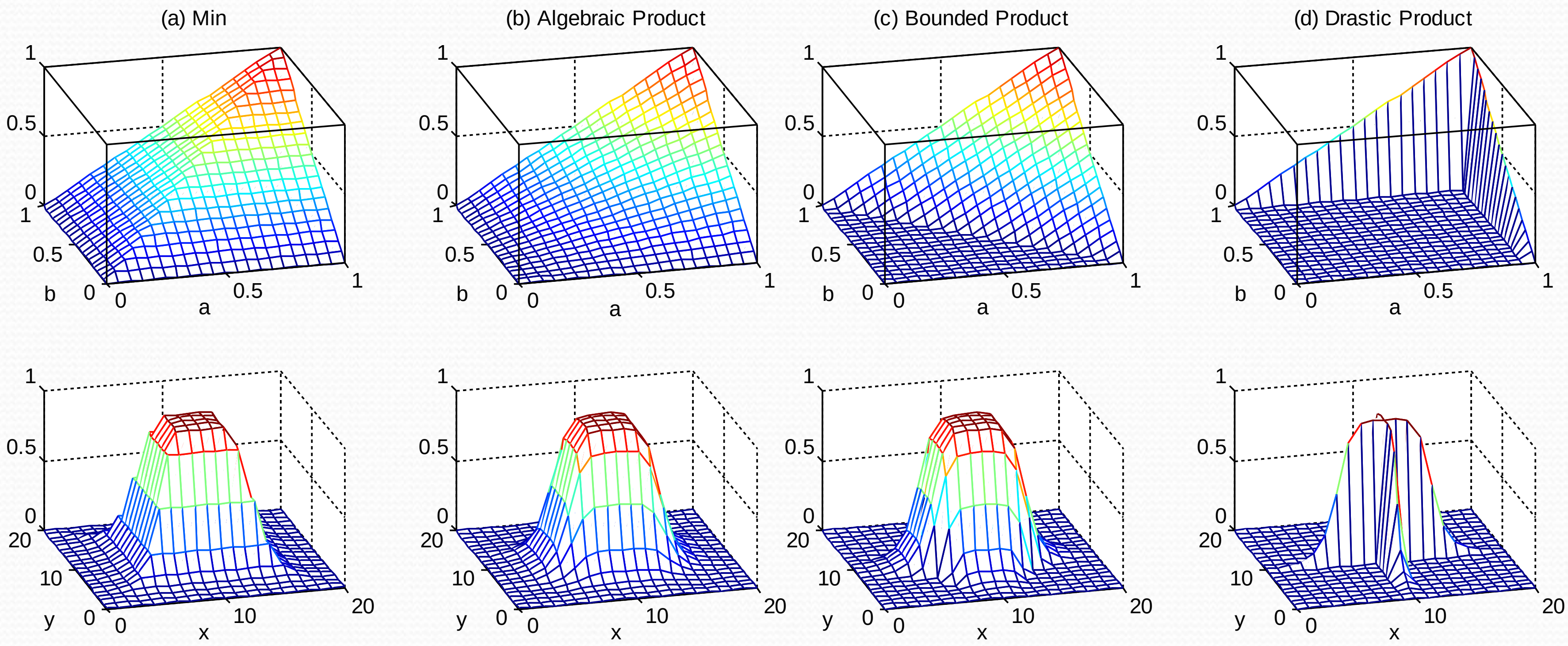




A coupled with B

Fuzzy implication function:

$$m_R(x,y) = f(m_A(x), m_B(y)) = f(a,b)$$



A entails B

- Boolean fuzzy implication (based on $\neg A \vee B$)

$$m_R(x, y) = max(1 - m_A(x), m_B(y))$$

- Zadeh's max-min implication (based on $\neg A \vee (A \wedge B)$)

$$m_R(x, y) = max(1 - m_A(x), min(m_A(x), m_B(y)))$$

- Zadeh's arithmetic implication (based on $\neg A \vee B$)

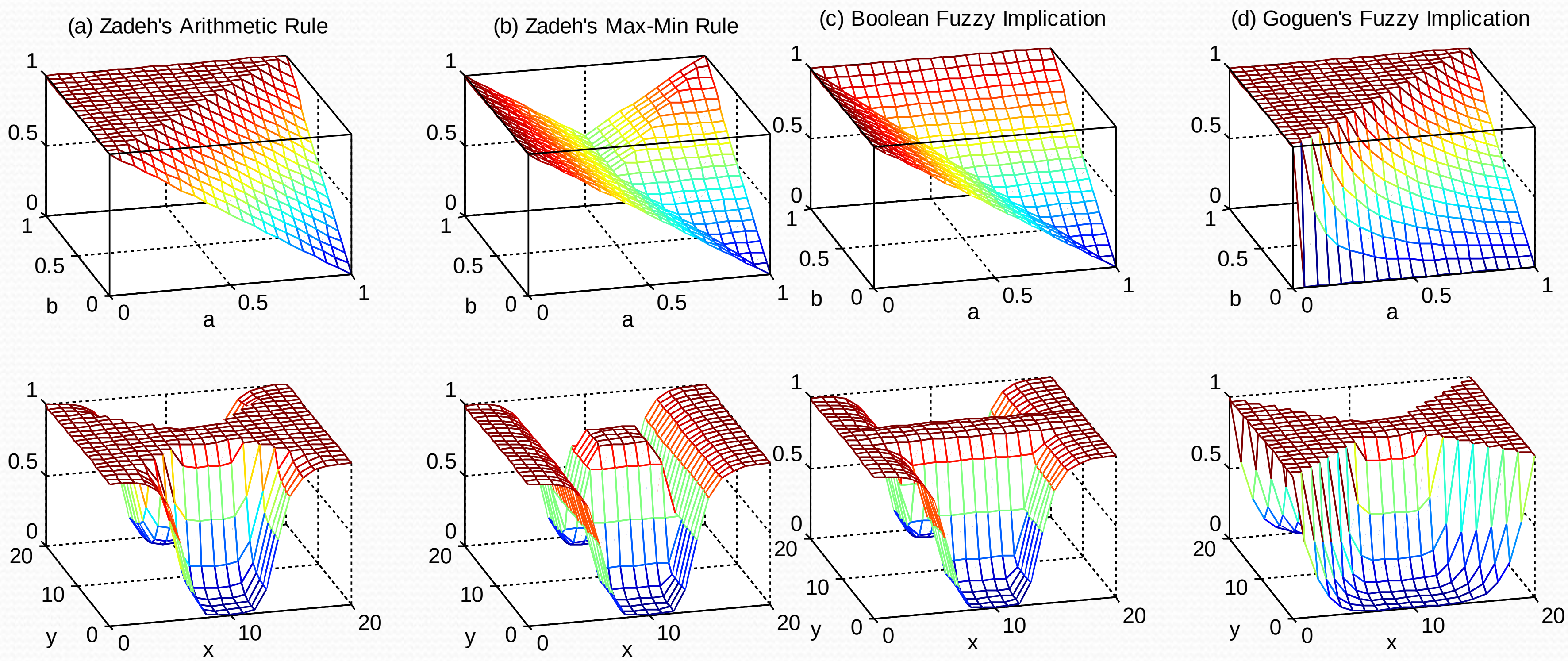
$$m_R(x, y) = min(1 - m_A(x) + m_B(y), 1)$$

- Goguen's implication

$$m_R(x, y) = min(m_B(x) / m_A(y), 1)$$

A entails B

Material implication



Constructing fuzzy relations

Premise: Young people make long GSM calls

X = {18, 20, 22, 25, 30} [years]

Y = {1, 3, 5, 7, 10, 20} [min./call]

young(x)

x	18	20	22	25	30
$\mu(x)$	1	1	0.8	0.5	0.2

long(y)

y	1	3	5	7	10	20
$\mu(y)$	0	0.1	0.2	0.5	0.9	1

Constructing fuzzy relations

Compute cylindrical extensions

young(x) into X x Y

x	$\mu(x)$	y = 1	3	5	7	10	20
18	1	1	1	1	1	1	1
20	1	1	1	1	1	1	1
22	0.8	0.8	0.8	0.8	0.8	0.8	0.8
25	0.5	0.5	0.5	0.5	0.5	0.5	0.5
30	0.2	0.2	0.2	0.2	0.2	0.2	0.2

long(y) into X x Y

y	1	3	5	7	10	20
$\mu(y)$	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1
	0	0.1	0.2	0.5	0.9	1

Constructing fuzzy relations

Compute the aggregation of the two cylindrical extensions. Since "young" and "long" go together, you can use a conjunctive operator (e.g. minimum).

1	1	1	1	1	1
1	1	1	1	1	1
0.8	0.8	0.8	0.8	0.8	0.8
0.5	0.5	0.5	0.5	0.5	0.5
0.2	0.2	0.2	0.2	0.2	0.2

0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1
0	0.1	0.2	0.5	0.9	1

min

y

x	1	3	5	7	10	20
18	0	0.1	0.2	0.5	0.9	1
20	0	0.1	0.2	0.5	0.9	1
22	0	0.1	0.2	0.5	0.8	0.8
25	0	0.1	0.2	0.5	0.5	0.5
30	0	0.1	0.2	0.2	0.2	0.2

R

Projection

x	y / $\mu(x) \backslash \mu(y)$	1 / 0	3 / 0.1	5 / 0.2	7 / 0.5	10 / 0.9	20 / 1	$\Pr_X(R)$
18	1	0	0.1	0.2	0.5	0.9	1	1
20	1	0	0.1	0.2	0.5	0.9	1	1
22	0.8	0	0.1	0.2	0.5	0.8	0.8	0.8
25	0.5	0	0.1	0.2	0.5	0.5	0.5	0.5
30	0.2	0	0.1	0.2	0.2	0.2	0.2	0.2
	$\Pr_Y(R)$	0	0.1	0.2	0.5	0.9	1	

young

long

Max-min composition

The max-min composition of two fuzzy relations *R* (defined on *X* and *Y*) and *S* (defined on *Y* and *Z*) is

$$\mu_{R \circ S}(x, z) = \bigvee_{y} [\mu_R(x, y) \wedge \mu_S(y, z)]$$

The result is the combined relation defined on X and Z

Max-min composition

example

People who make long GSM calls give a lot of money to clothing

Young people give a lot of money to clothing

$$\begin{bmatrix} 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\ 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1 \\ 0 & 0.1 & 0.2 & 0.5 & 0.8 & 0.8 \\ 0 & 0.1 & 0.2 & 0.5 & 0.5 & 0.5 \\ 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2 \end{bmatrix} \circ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.1 & 0.1 \\ 0 & 0.2 & 0.2 \\ 0 & 0.5 & 0.5 \\ 0 & 0.5 & 0.9 \\ 0 & 0.5 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0.5 & 1 \\ 0 & 0.5 & 1 \\ 0 & 0.5 & 0.8 \\ 0 & 0.5 & 0.5 \\ 0 & 0.2 & 0.2 \end{bmatrix}$$

R **S** **R°S**

Compositional rule of inference

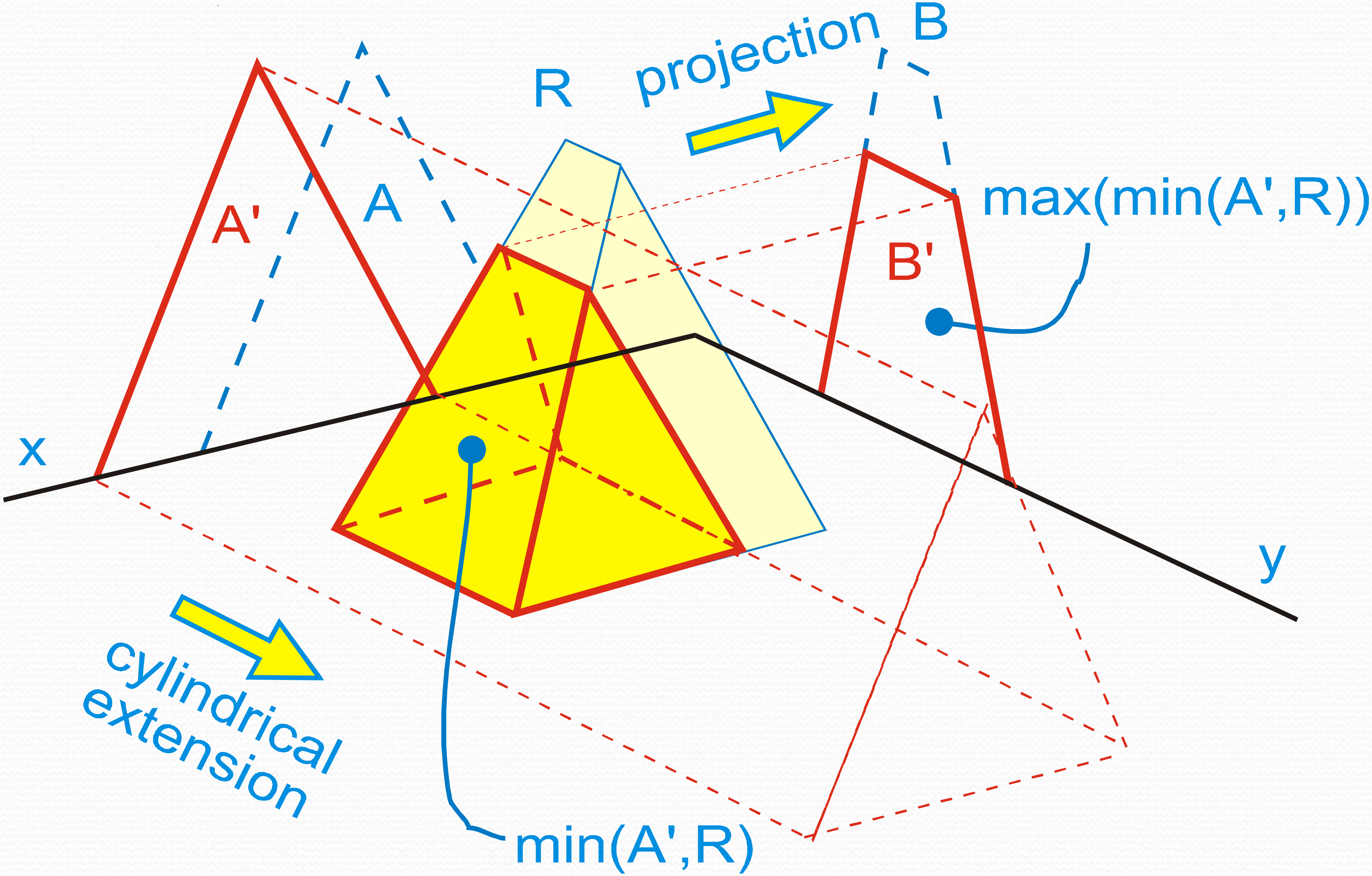
Max-min composition of a fuzzy set A on X and a fuzzy relation R on X x Y

$$\mu_{A \circ R}(y) = \max_{x \in X} \{\mu_A(x) \wedge \mu_R(x, y)\} \ \forall y \in Y$$

Returns the image of A transformed through the relation R

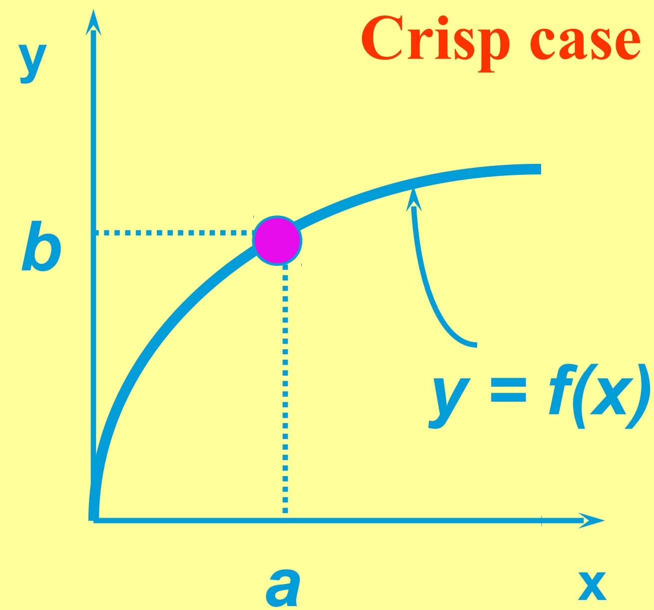
The composition of A and R can also be seen as the shadow of R on A (for every y)

Compositional rule of inference



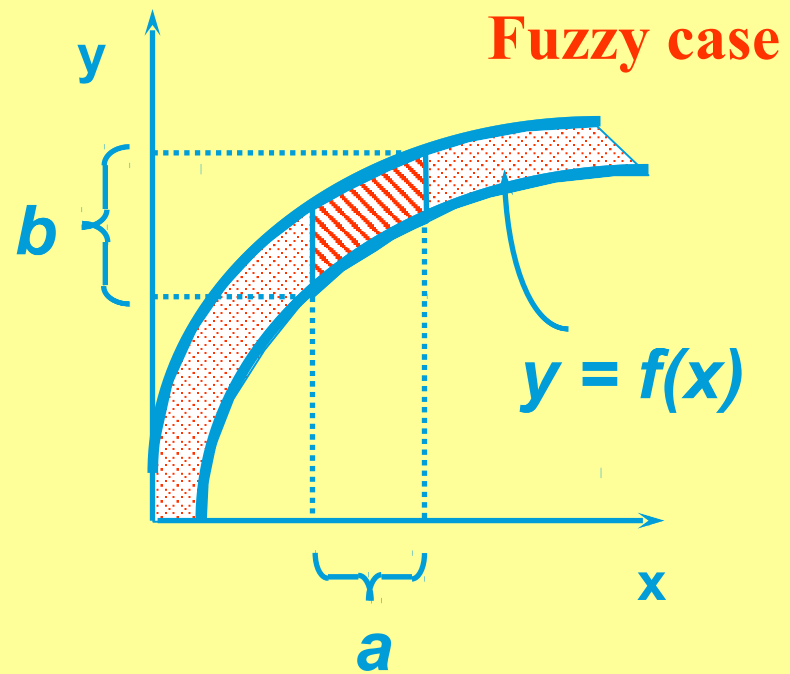
Compositional rule of inference

Derivation of $y = b$ from $x = a$ and $y = f(x)$:



a **and** ***b*****: points**

y = f(x) **: a curve**

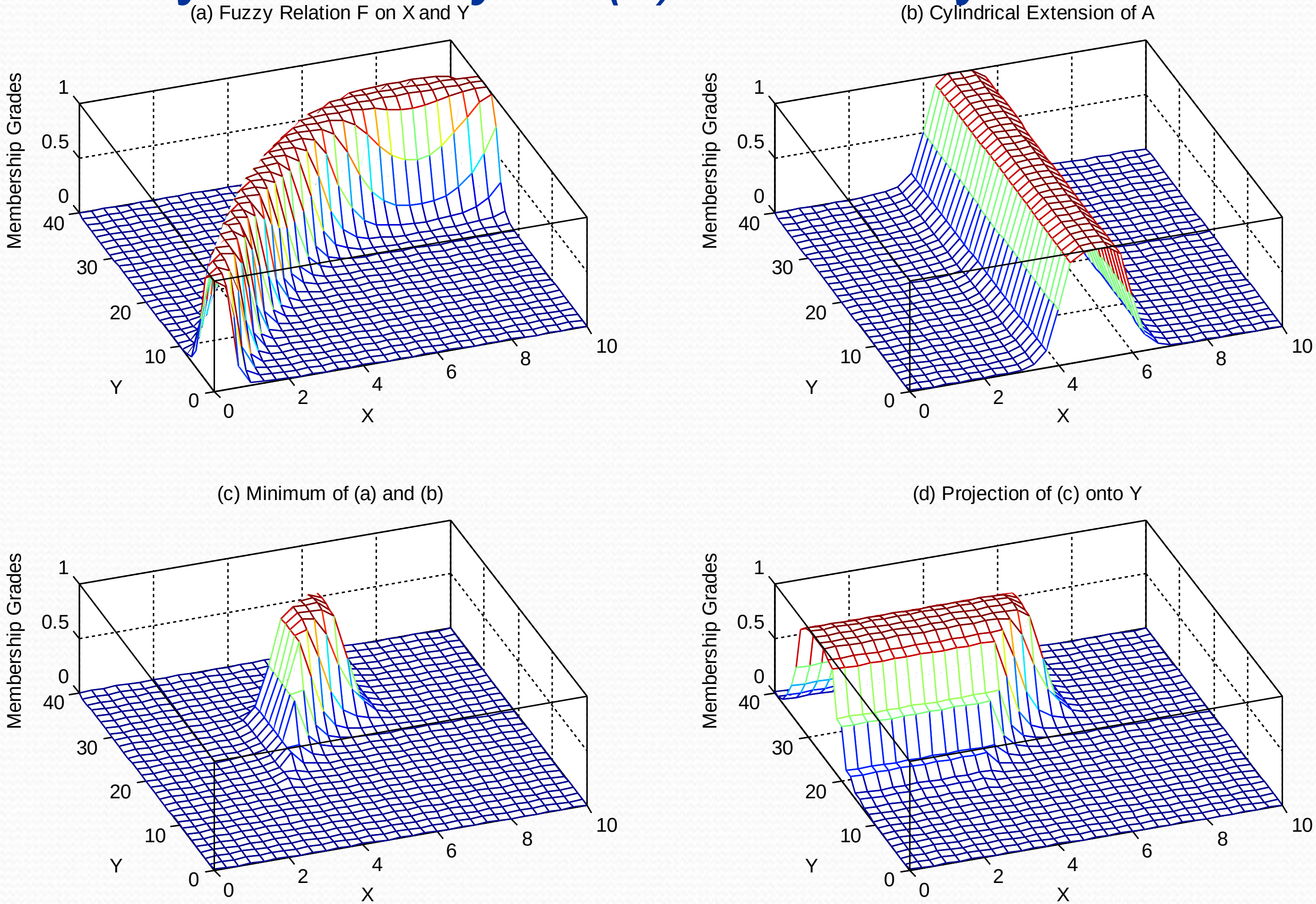


a **and** ***b*****: intervals**

y = f(x) **: an interval-valued function**

Compositional rule of inference

a is a fuzzy set and *y* = *f(x)* is a fuzzy relation

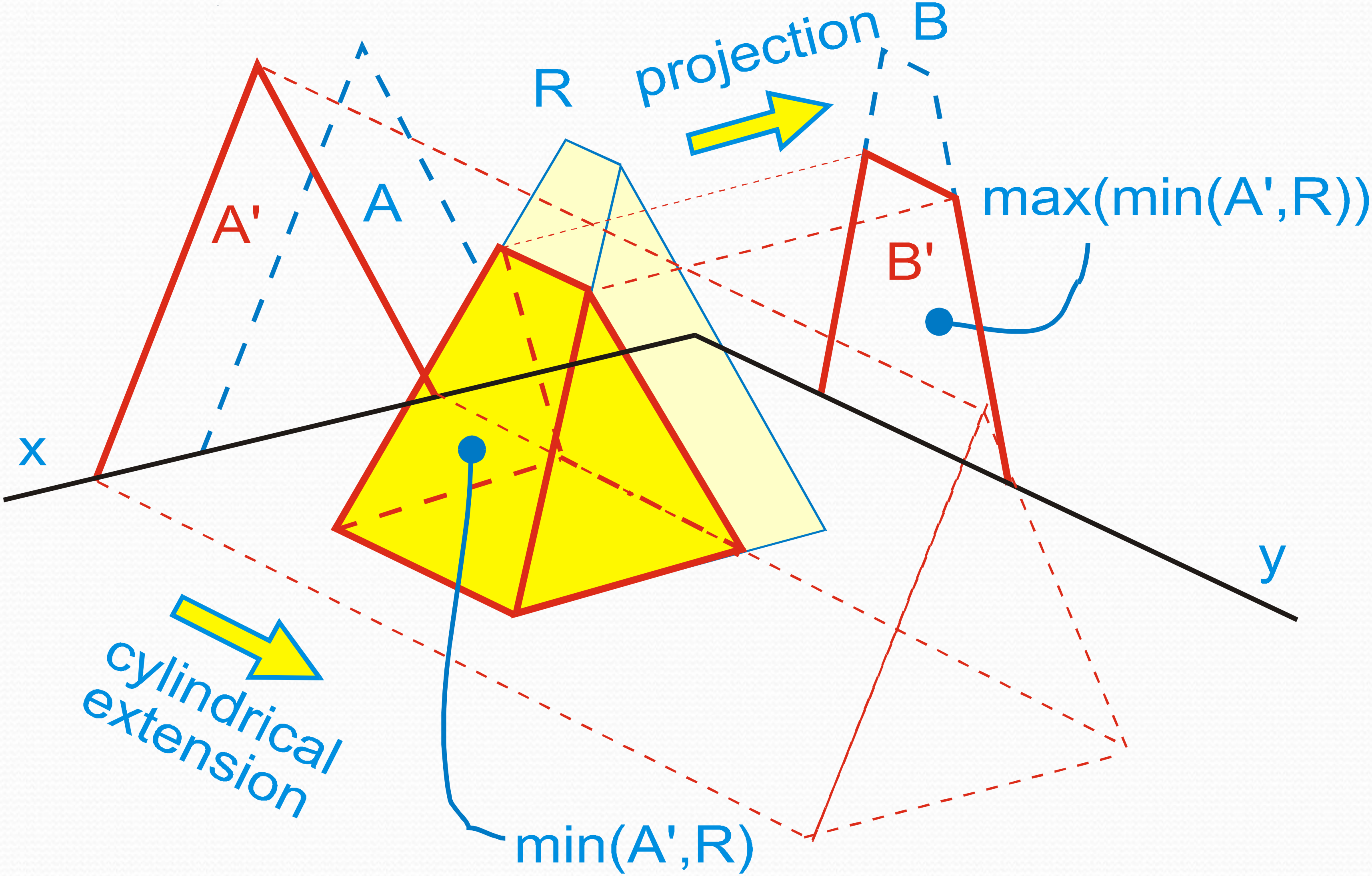


CRI example (discrete)

$$\begin{bmatrix}0 & 0 & 0 & 1 & 0\end{bmatrix} \circ \begin{bmatrix}0 & 0.1 & 0.2 & 0.5 & 0.9 & 1\\ 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1\\ 0 & 0.1 & 0.2 & 0.5 & 0.8 & 0.8\\ 0 & 0.1 & 0.2 & 0.5 & 0.5 & 0.5\\ 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2\end{bmatrix} = \begin{bmatrix}0 & 0.1 & 0.2 & 0.5 & 0.5 & 0.5\end{bmatrix}$$

$$\begin{bmatrix}0 & 0.4 & 1 & 0.2 & 0\end{bmatrix} \circ \begin{bmatrix}0 & 0.1 & 0.2 & 0.5 & 0.9 & 1\\ 0 & 0.1 & 0.2 & 0.5 & 0.9 & 1\\ 0 & 0.1 & 0.2 & 0.5 & 0.8 & 0.8\\ 0 & 0.1 & 0.2 & 0.5 & 0.5 & 0.5\\ 0 & 0.1 & 0.2 & 0.2 & 0.2 & 0.2\end{bmatrix} = \begin{bmatrix}0 & 0.1 & 0.2 & 0.5 & 0.8 & 0.8\end{bmatrix}$$

Inference: Coupled Memberships

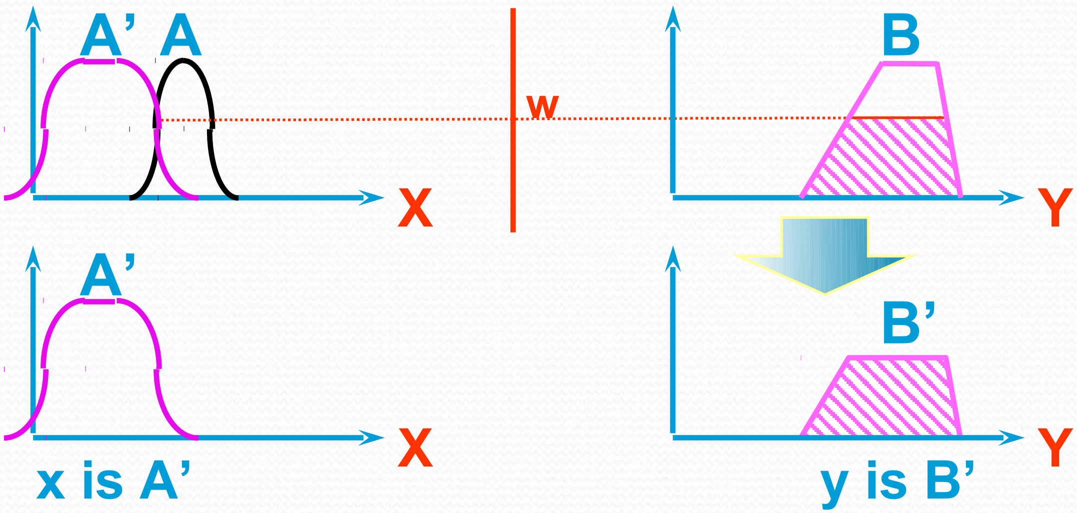


Single rule, single antecedent

- Rule: if x is A then y is B
- Fact: x is A′
- Conclusion: y is B′

$$\mu_{B'}(y) = [\vee_x(\mu_{A'}(x) \wedge \mu_A(x))] \wedge \mu_B(y)$$
$$= w \wedge \mu_B(y)$$

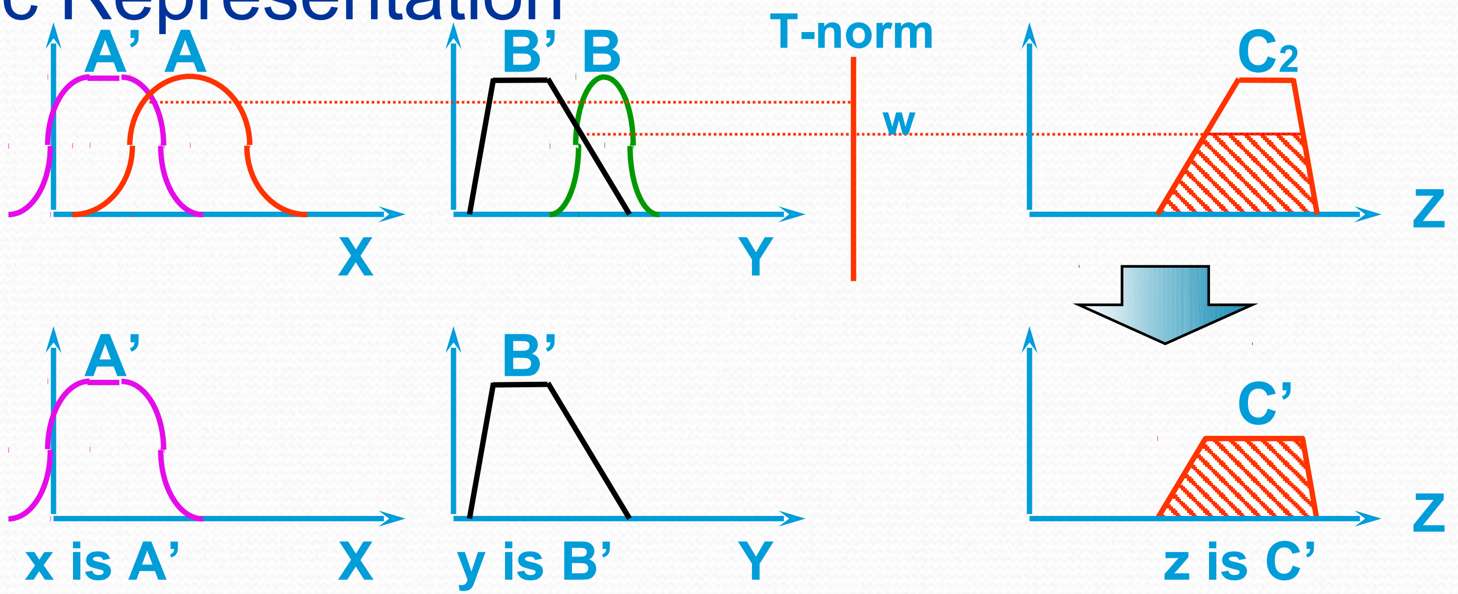
Graphic Representation



Single rule, multiple antecedents

- Rule: if x is A and y is B then z is C
- Fact: x is A′ and y is B′
- Conclusion: z is C′

Graphic Representation

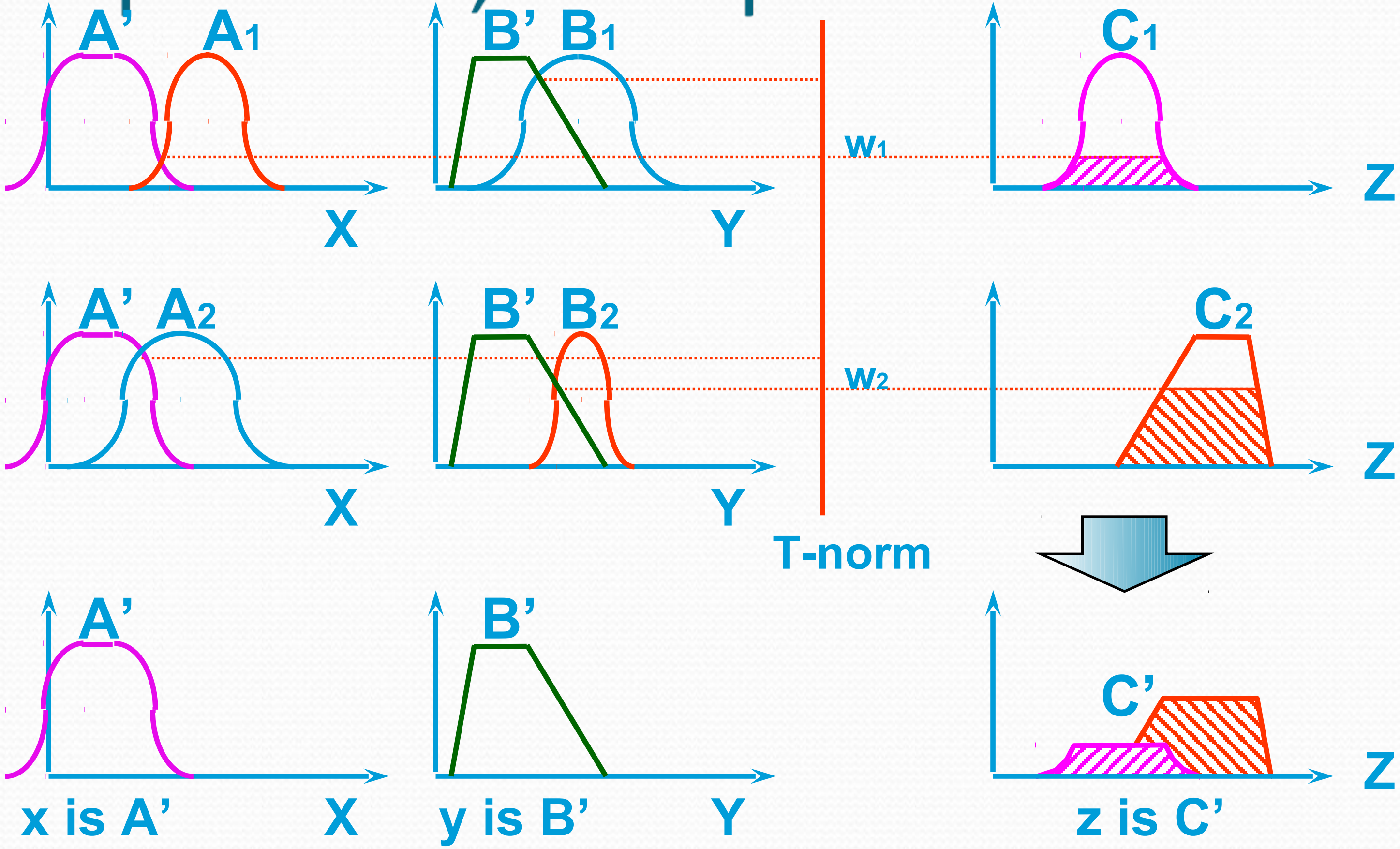


Multiple Rules

- Fuzzy rules like A → B are represented as fuzzy relation
- The collection of all rules is represented as an aggregated relation using the union operator, i.e.

$$R_{\text{tot}} = \bigcup_{i=1}^{N} R_i = \bigvee_{i=1}^{N} R_i$$

Multiple rules, multiple antecedents



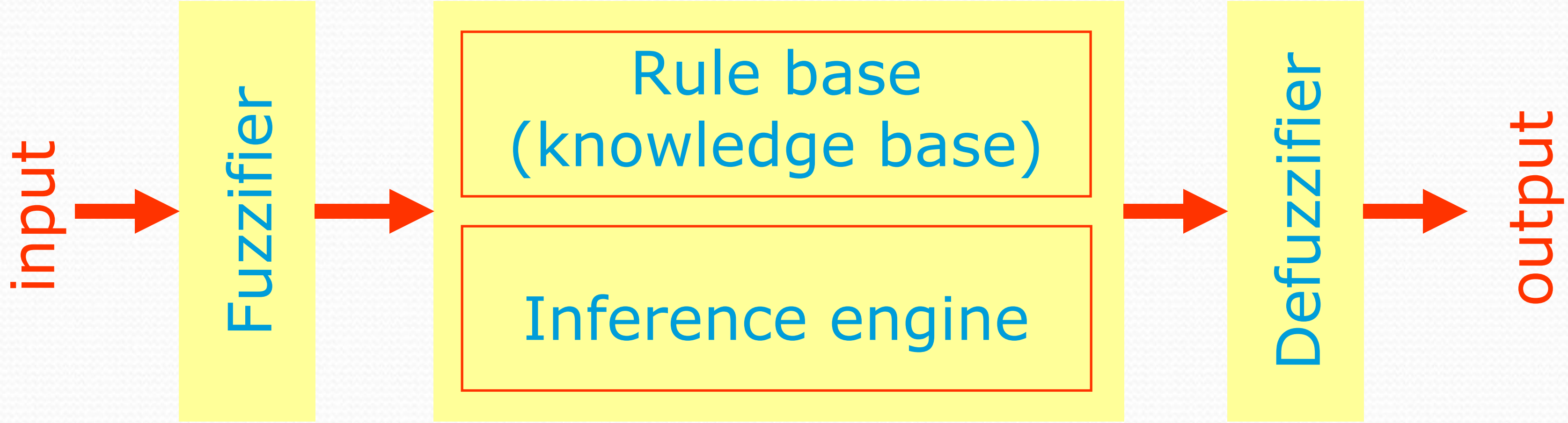
Fuzzy inference system

Multiple names

- Fuzzy rule-based system
- Fuzzy expert system
- Fuzzy model
- Fuzzy associative memory
- Fuzzy logic controller
- Fuzzy system (simply)

Building blocks

- Fuzzifier
- Rule base
- Inference engine
- Defuzzifier



Fuzzifier

- Interface between the outside world and the fuzzy system (input)
- Transform a measurement in a fuzzy set
- Multiple methods:
  - singleton method
  - triangular method
  - Gaussian function method
  - …

Fuzzification

- singleton method

  $m_A(x) = 1$ when $x = a$, otherwise 0

  X = {a,b,c,d}

  a → { (a,1), (b,0), (c,0), (d,0) }

- triangular function method

  $m_A(x) = max(0, |x-a|/s)$

  (Here a is the measurement)

Rule base

- Heart of the knowledge base of the fuzzy system
- Encodes the general relation between the inputs and the outputs
- Rules can be examples, rules of thumb, encoded experience, qualitative relations between variables, etc.
- Rules are often represented as if-then statements

Inference engine

- The reasoning mechanism of the fuzzy system (to infer: to reason/deduce)
- Combines actual inputs with the information encoded in the rule base to compute the fuzzy output of the system
- Usually implements the *compositional rule of inference* or some equivalent computation
- Not as context-independent as the inference engine of an expert system

Defuzzifier

- Interface between the fuzzy systems and the outside world (output)
- Needed when a crisp output is required (e.g. a final decision, a control action, a final advice, etc.)
- Computes a number/symbol that represents the output fuzzy set
- Enhances the interpolation properties of the fuzzy system

Mamdani fuzzy models

Five major steps:

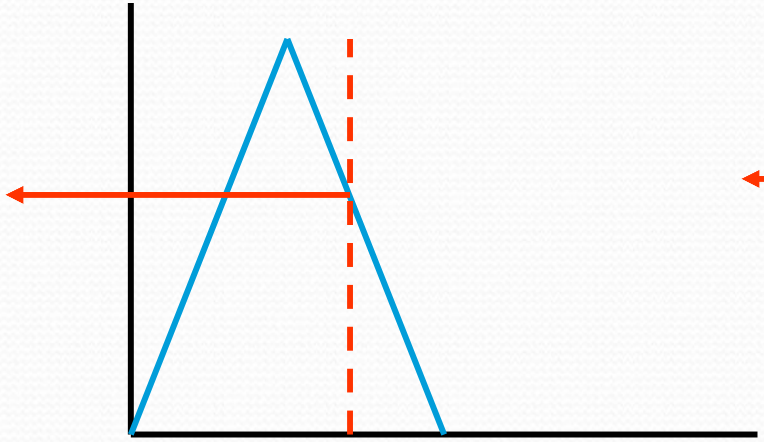
- Fuzzification
- Degree of fulfillment
- Inference
- Aggregation
- Defuzzification

Computations from Mamdani reasoning are mathematically equivalent to the compositional rule of inference

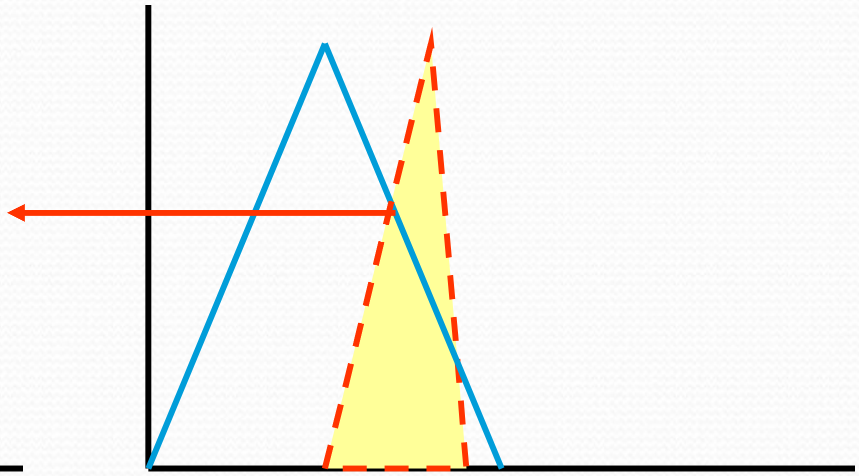
Fuzzification

Calculate the membership degrees of the (measured) inputs to the linguistic terms in the fuzzifier

Crisp inputs



Fuzzy inputs



Degree of fulfillment

When the antecedent (if-part) of a rule contains multiple variables, the total match between all inputs and the rule antecedent must be determined

Degree of fulfilment determines to what degree the rule is valid

Take minimum of all fuzzified values in the rule antecedent

Inference

Computes the output of each fuzzy rule in the rule base, given the degree of fulfilment of the rule base

Modifies the output fuzzy set depending on the degree of fulfilment

Clip the output fuzzy set at the height corresponding to the degree of fulfillment

Aggregation

Multiple rules may become active in a fuzzy system

Combined output of the fuzzy system is the combined output of all active rules

Corresponds to calculating the total relation from the relations of individual rules

Calculate maximum of individual fuzzy rule outputs

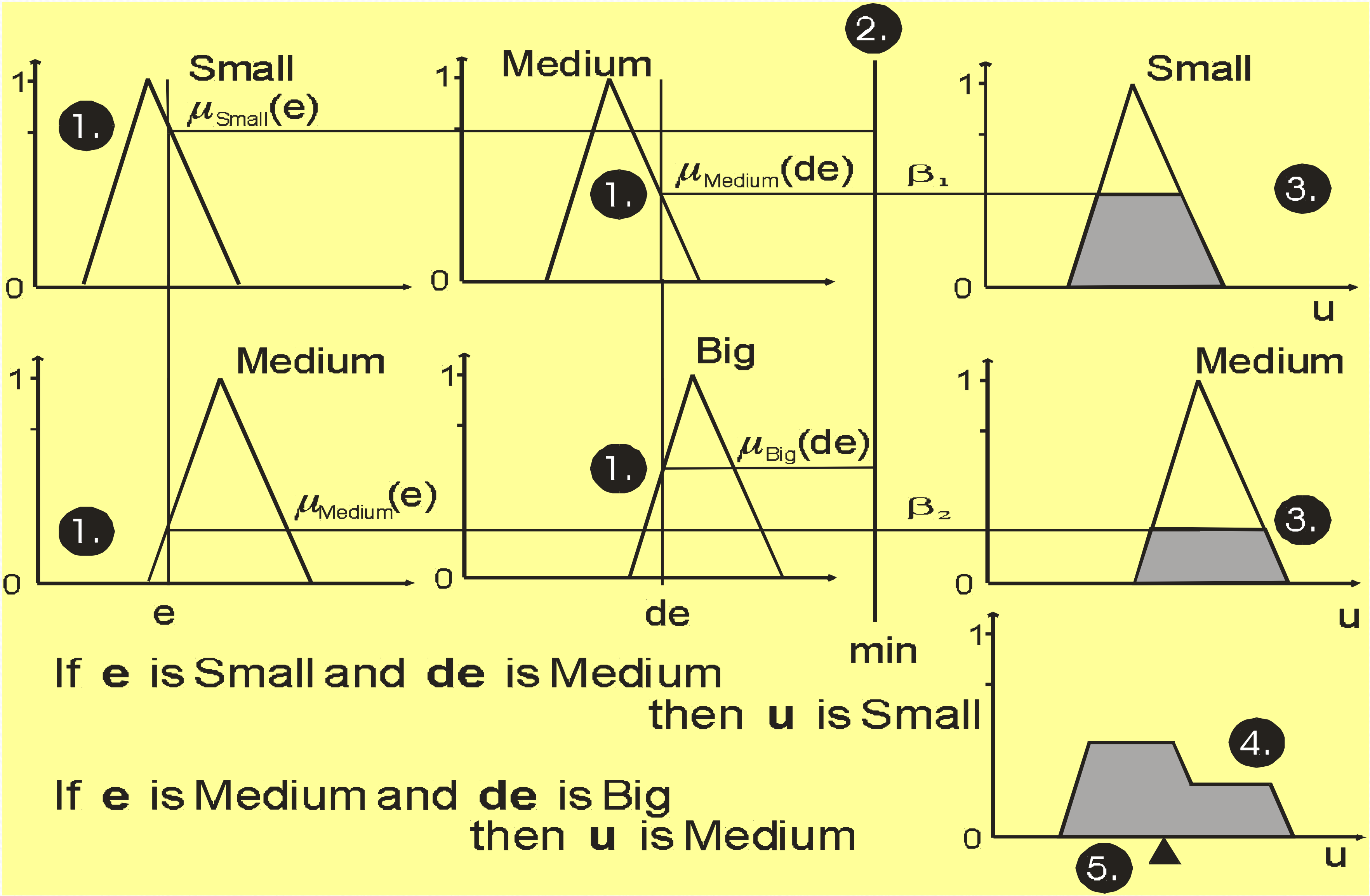
Defuzzification

Determines the crisp output of the fuzzy system

Replaces the output fuzzy set with a representative number

Compute the defuzzified value using an agreed upon defuzzification operator (e.g. center of area)

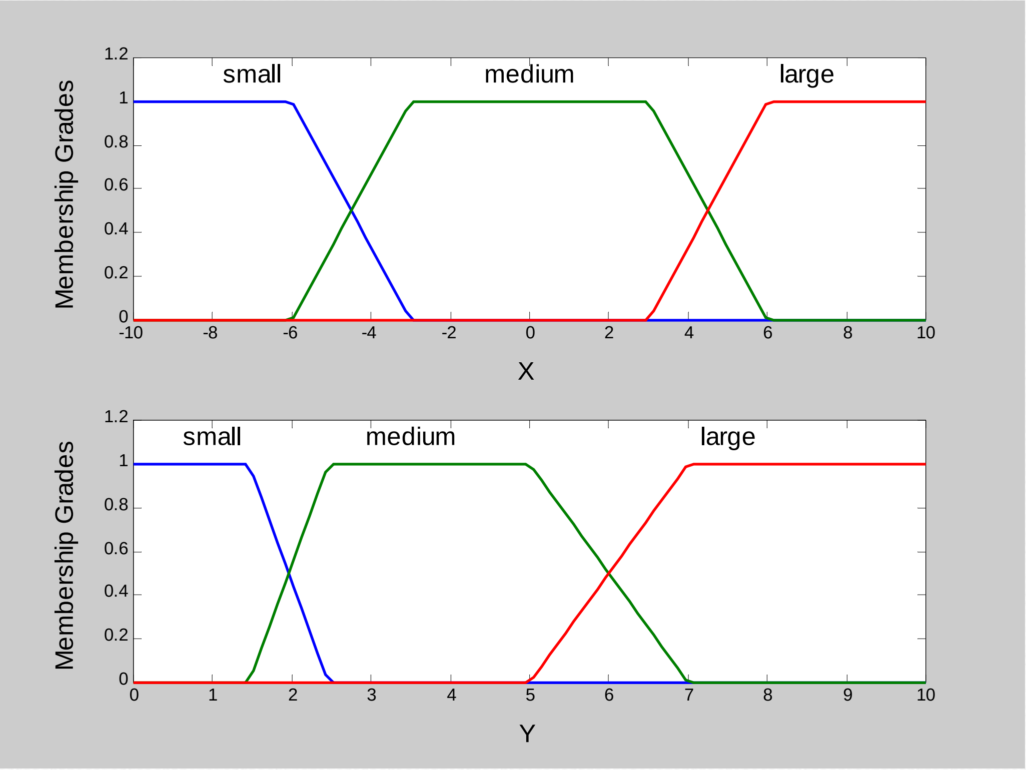
Mamdani system - example



Defuzzification rules

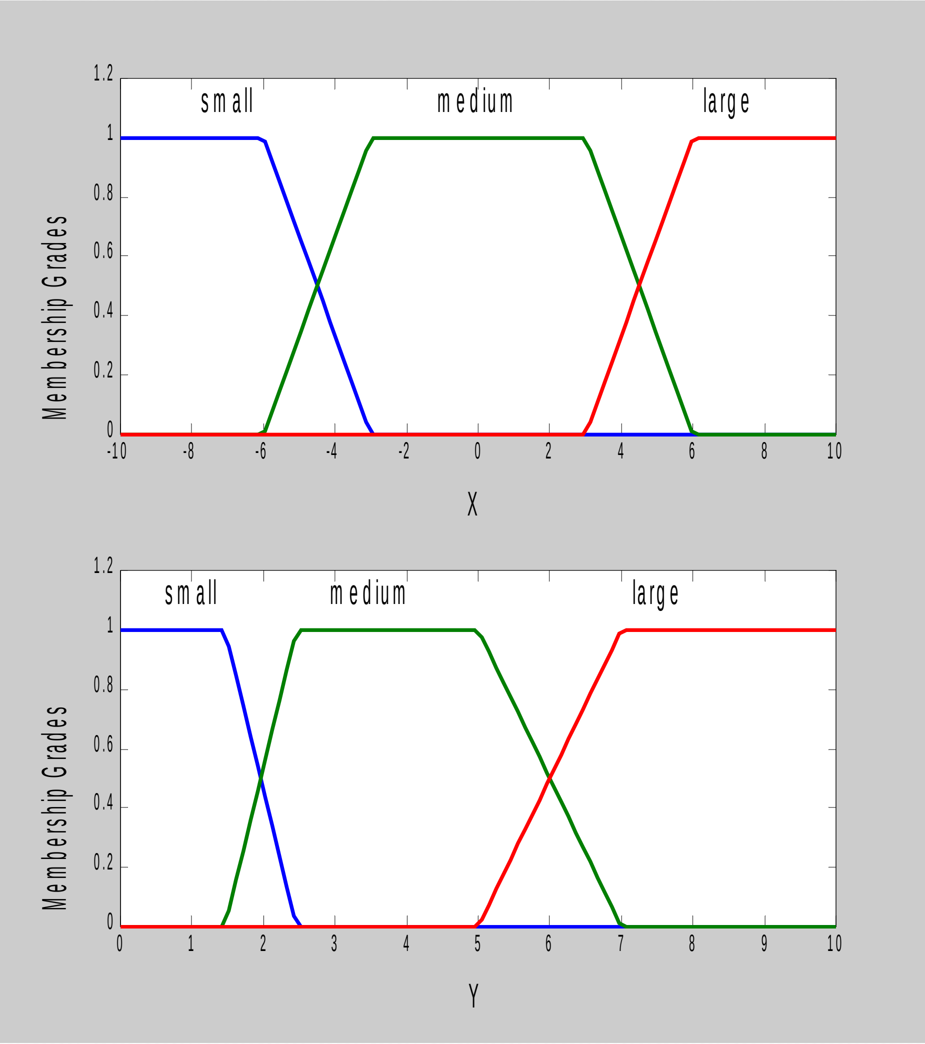
- Centroid-of-area $$z^* = \frac{\int_Z \mu_A(z)\, z\, dz}{\int_Z \mu_A(z)\, dz}$$
- Bisector of area $$\int_{-\infty}^{z^*} \mu_A(z) dz = \int_{z^*}^{\infty} \mu_A(z) dz$$
- Mean of maximum $$z^* = \frac{\int_{Z'} z\, dz}{\int_{Z'} dz}, \quad Z' = \{z \mid \mu_A(z) = \mu^*\}$$
- Smallest of maximum $$\min_{z \in Z'} z$$
- Largest of maximum $$\max_{z \in Z'} z$$

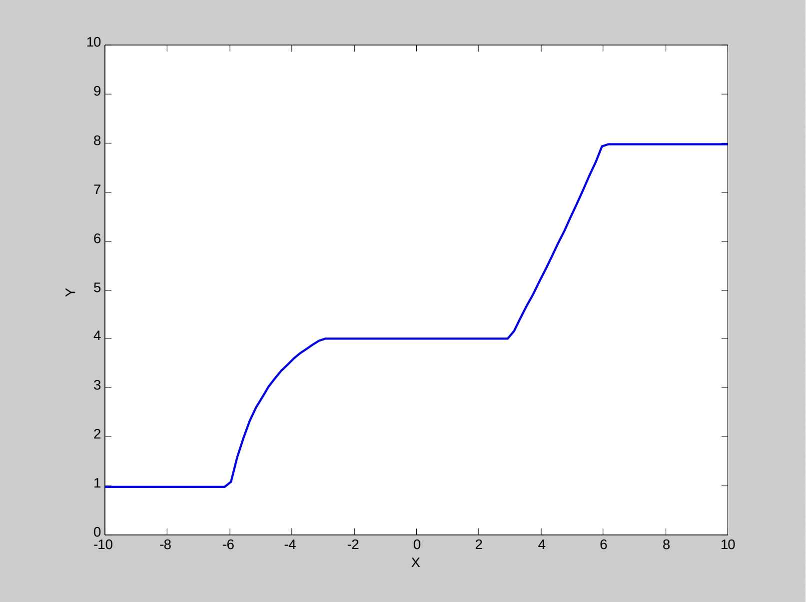
Mamdani - single input



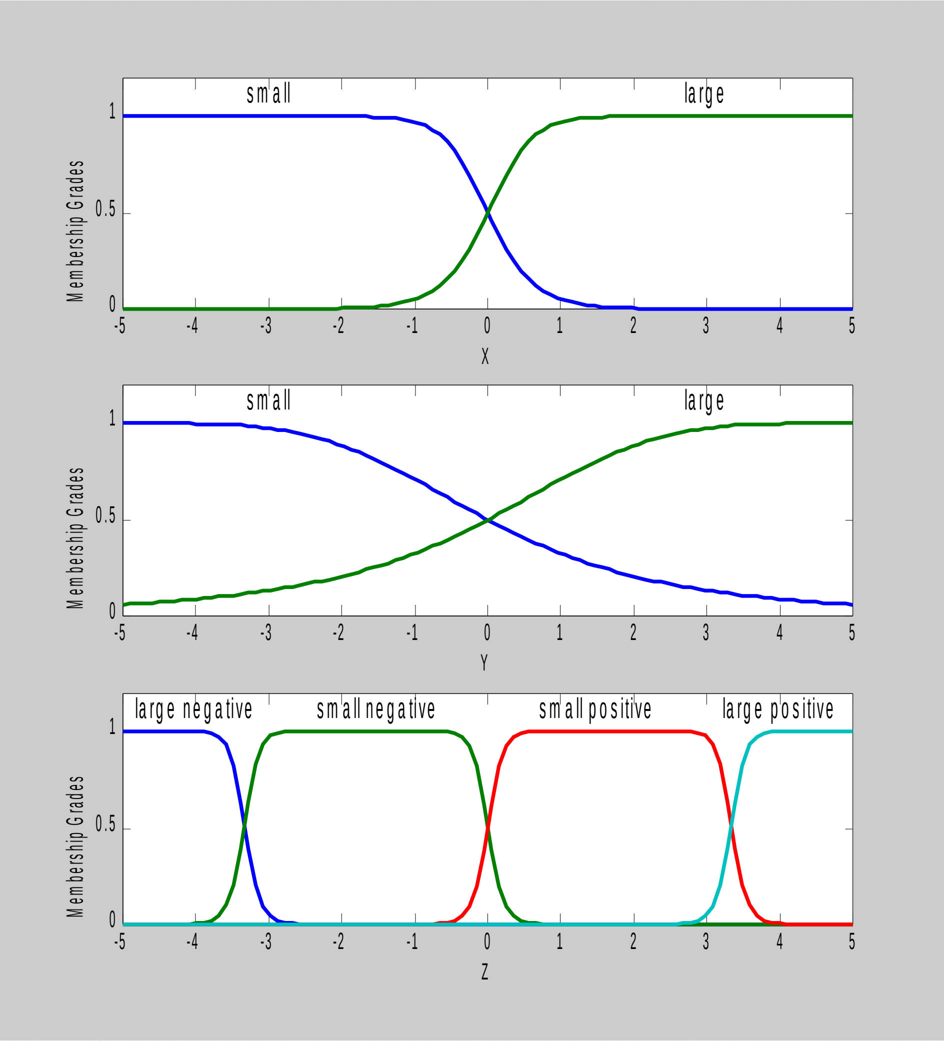
- X is Small → Y is Small
- X is Medium → Y is Medium
- X is Large → Y is Large

Mamdani - single input



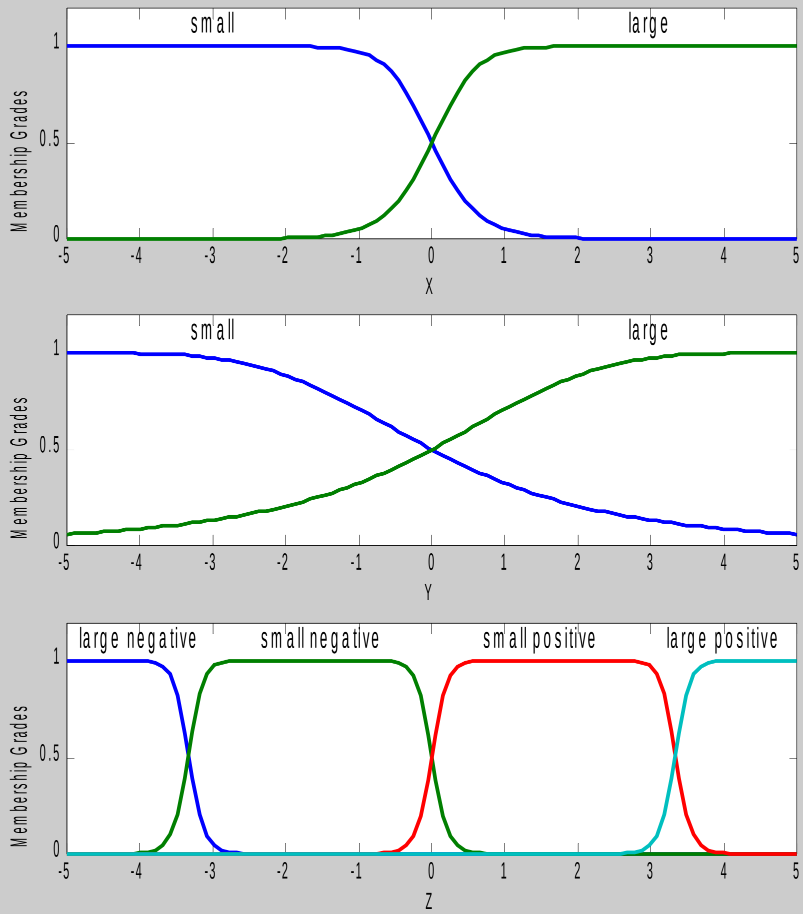


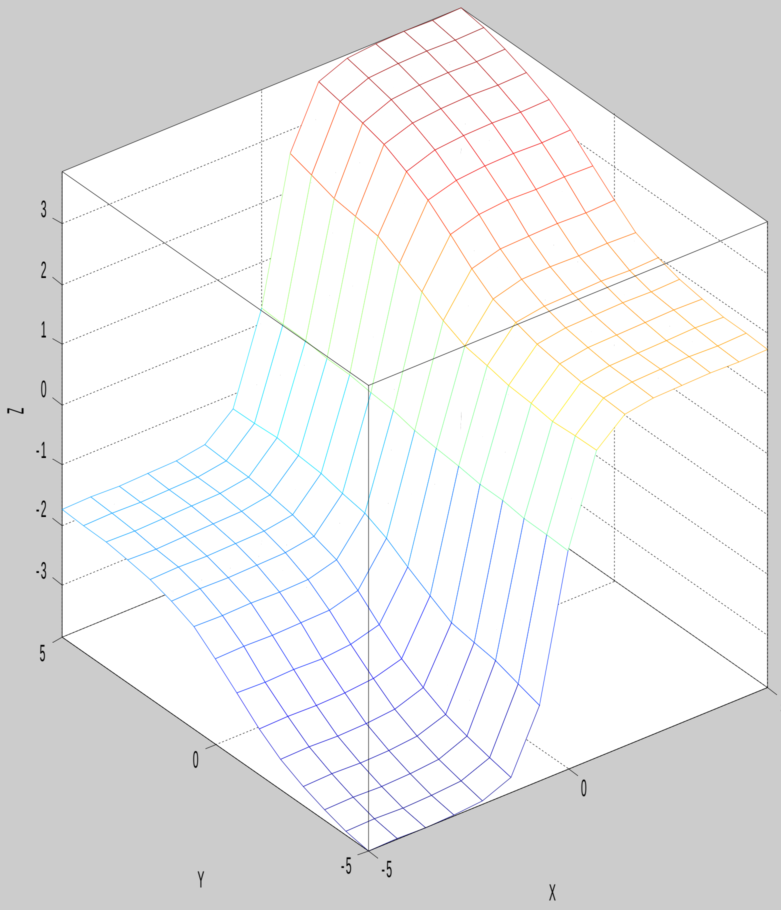
Mamdani - double input



- X is Small and Y is Small → Z is negative Large
- X is Small and Y is Large → Z is negative Small
- X is Large and Y is Small → Z is positive Small
- if X is Large and Y is Large → Z is positive Large

Mamdani - double input



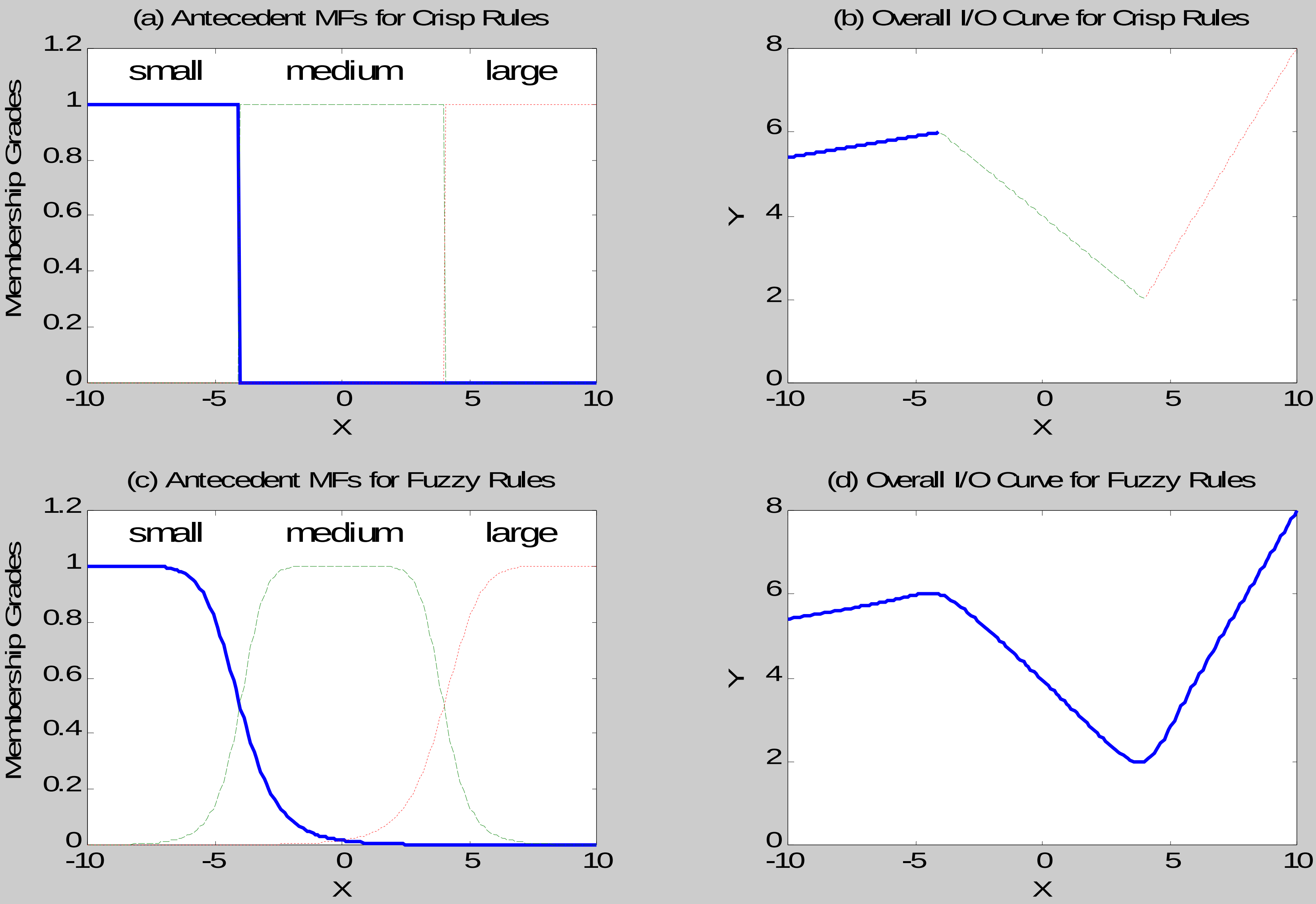


Takagi-Sugeno fuzzy models

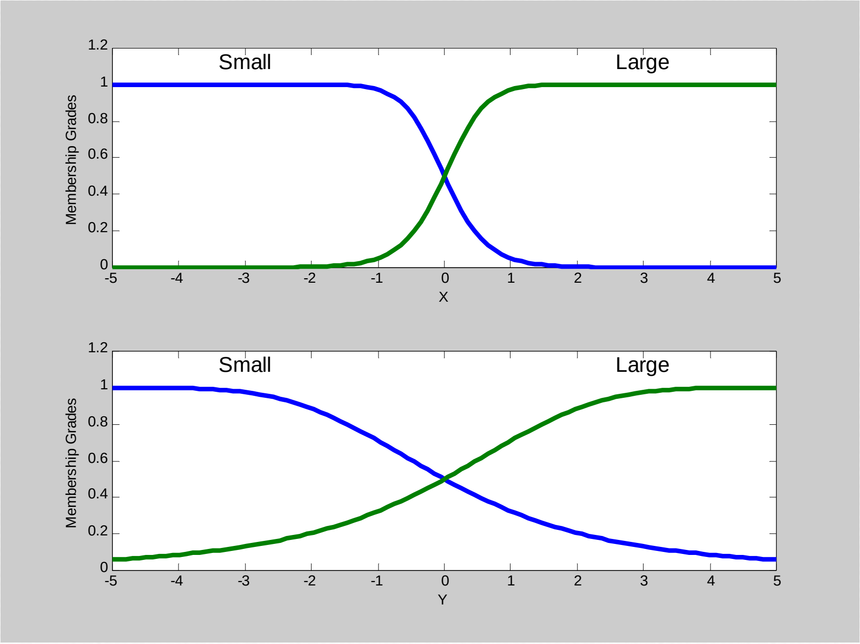
- Fuzzy antecedents, crisp consequents
- Consequent is a crisp function of inputs
  if x is A and y is B then z = f(x,y)
- Zero-order Sugeno: constant consequent
  if x is A and y is B then z = c
- First-order Sugeno: linear consequent
  if x is A and y is B then z = ax+by+c
- Overall output is a weighted average of individual rule outputs

$$z^* = \frac{\sum_{i=1}^{N} \beta_i \, z_i}{\sum_{i=1}^{N} \beta_i}$$

Fuzzy vs. crisp rule set

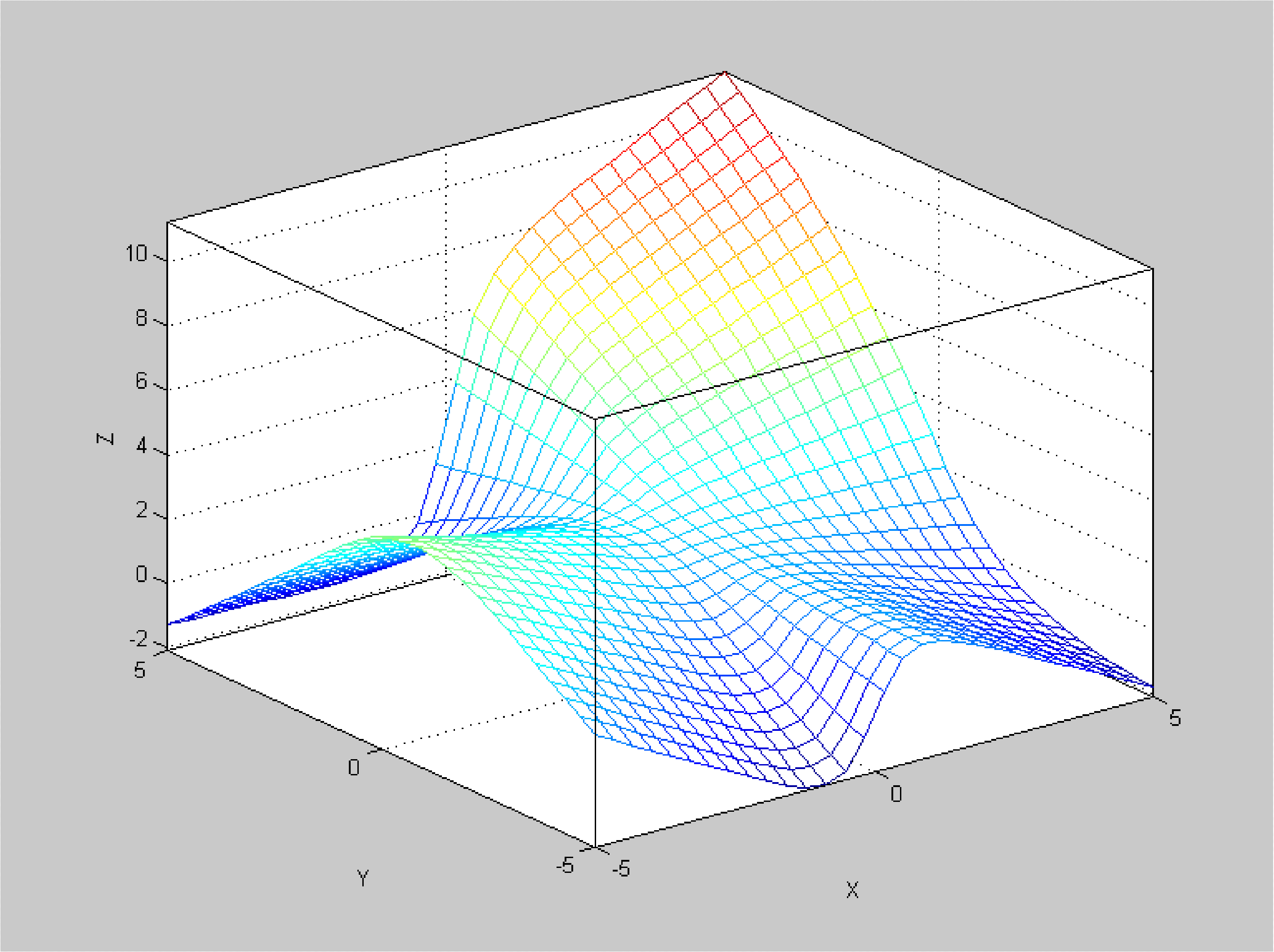


Sugeno - double input



- If X is Small and Y is Small then $z = -x+y+1$
- If X is Small and Y is Large then $z = -y+3$
- If X is Large and Y is Small then $z = -x+3$
- If X is Large and Y is Large then $z = x+y+2$

Sugeno - double input



Approximation capability

- Fuzzy systems are general function approximators (c.f. neural networks)
- You can increase the accuracy of a mapping by increasing the number of rules (examples) in the rule base
- Best results are obtained when the number of linguistic terms in the input and the output are increased (a finer partition)
- Using too many linguistic terms diminishes the transparency of fuzzy systems

Interpolation properties

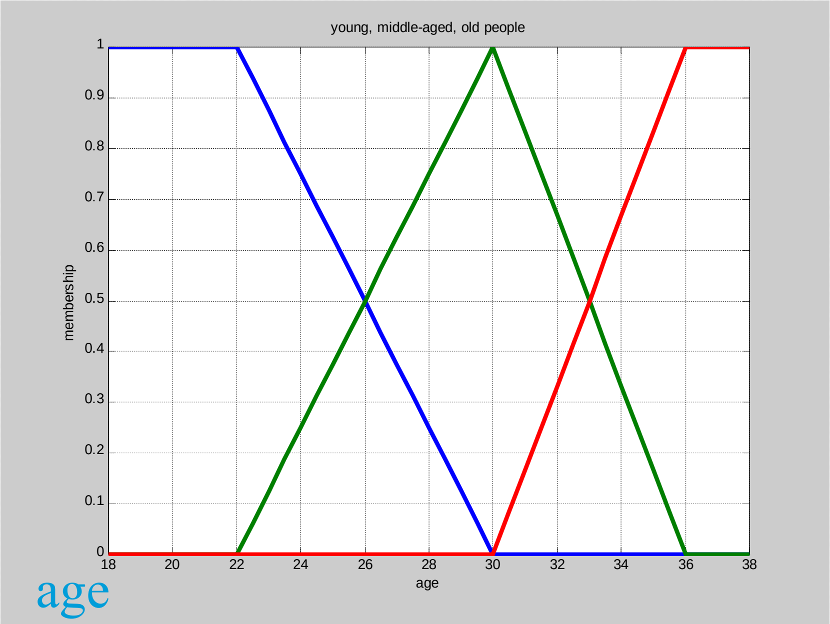
- Multiple rules in a fuzzy system may fire (become active) because fuzzy sets overlap
- Fuzzy rules represent typical cases or examples of the relation between two quantities
- The reasoning mechanism interpolates between the rules to determine the system output

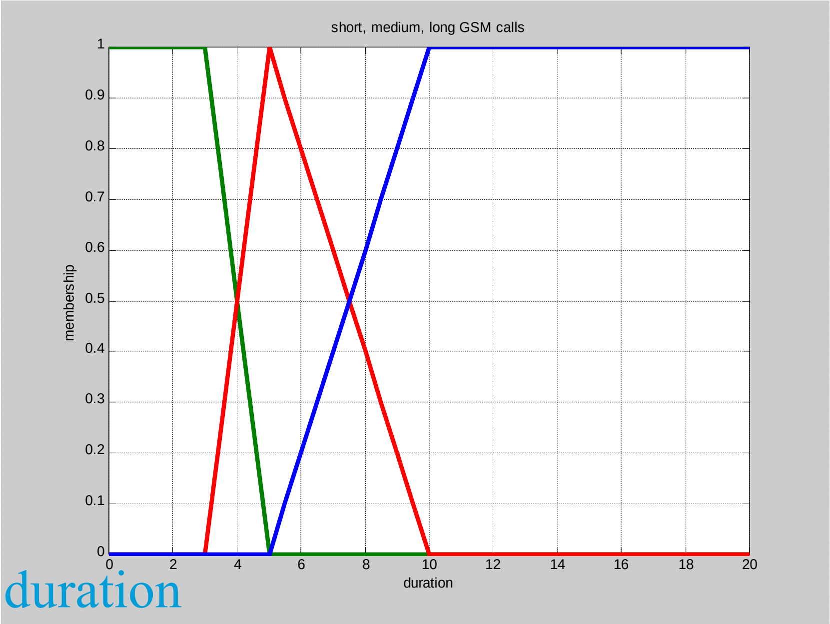
Interpolation properties

example

Consider a fuzzy system with three rules

- Young people make long GSM calls
- Middle aged people make short GSM calls
- Old people make medium-long GSM calls





Interpolation properties

example